Math 217 Fall 2025
Quiz 36 – Solutions

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1. Complete* the partial sentences below into precise definitions for, or precise mathematical characterizations of, the italicized term:

   (a) A square matrix A is *orthogonally diagonalizable* if ...

   > **Solution:** A square matrix A is *orthogonally diagonalizable* if there exists an orthogonal matrix Q (so $Q^TQ = I$) and a diagonal matrix D such that
   > $$Q^TAQ = D.$$

   (b) An *elementary matrix* is ...

   > **Solution:** An *elementary matrix* is an $n \times n$ matrix obtained by performing a single elementary row operation on the identity matrix I_n.

   (c) Suppose U is a vector space and $u_1, u_2, \ldots, u_n \in U$. The list $(u_1, u_2, \ldots, u_n)$ is *linearly independent* if ...

   > **Solution:** The list $(u_1, u_2, \ldots, u_n)$ is *linearly independent* if whenever
   > $$a_1u_1 + a_2u_2 + \cdots + a_nu_n = 0$$
   > for scalars $a_1, \ldots, a_n$, it follows that
   > $$a_1 = a_2 = \cdots = a_n = 0.$$

2. Suppose V is a finite dimensional vector space and $T\colon V \to V$ is a linear transformation. Show: if $\chi_T(\lambda) = 0$, then
$$1 \leq \operatorname{gemu}(\lambda) \leq \operatorname{almu}(\lambda).$$

> **Solution:** If $\chi_T(\lambda) = 0$, then λ is a root of the characteristic polynomial of T, so
> $$\det(T - \lambda I) = 0.$$
> Thus $T - \lambda I$ is not invertible, so its kernel is nontrivial:
> $$\ker(T - \lambda I) \neq \{0\}.$$

*For full credit, please write out fully what you mean instead of using shorthand phrases.

Hence there exists a nonzero vector v with $(T - \lambda I)v = 0$, i.e. $T(v) = \lambda v$. Therefore the eigenspace

$$E_\lambda = \ker(T - \lambda I)$$

is nonzero and

$$\mathrm{gemu}(\lambda) = \dim(E_\lambda) \geq 1.$$

For each eigenvalue λ we always have

$$\mathrm{gemu}(\lambda) \leq \mathrm{almu}(\lambda),$$

where $\mathrm{almu}(\lambda)$ is the multiplicity of λ as a root of $\chi_T(x)$.

Combining these observations gives

$$1 \leq \mathrm{gemu}(\lambda) \leq \mathrm{almu}(\lambda).$$

3. If A is a symmetric $n \times n$ matrix and $U \subset \mathbb{R}^n$ is a subspace such that $A[U] \subset U$, then $A[U^\perp] \subset U^\perp$.

Solution: Since A is symmetric, we have

$$(Av) \cdot u = v \cdot (Au) \quad \text{for all } u, v \in \mathbb{R}^n.$$

Let $v \in U^\perp$. We want to show $Av \in U^\perp$, i.e. $(Av) \cdot u = 0$ for all $u \in U$.

Take any $u \in U$. By assumption, $Au \in U$. Since $v \in U^\perp$, we have

$$v \cdot (Au) = 0.$$

Using symmetry of A,

$$(Av) \cdot u = v \cdot (Au) = 0.$$

Thus Av is orthogonal to every vector in U, so $Av \in U^\perp$. Hence $A[U^\perp] \subset U^\perp$.

4. True or False. If you answer true, then state TRUE. If you answer false, then state FALSE. Justify your answer with either a short proof or an explicit counterexample.

   (a) Every eigenvalue of a symmetric matrix is a real number.

Solution: TRUE.

Let A be a real symmetric $n \times n$ matrix and let $v \neq 0$ be an eigenvector with eigenvalue λ, so $Av = \lambda v$.

Take the dot product of both sides with v:

$$(Av) \cdot v = (\lambda v) \cdot v = \lambda (v \cdot v).$$

Using symmetry of A,

$$(Av) \cdot v = v \cdot (Av) = v \cdot (\lambda v) = \overline{\lambda}\, (v \cdot v),$$

since the scalar exits on the second slot as a complex conjugate.
Thus,

$$\lambda(v \cdot v) = \overline{\lambda}(v \cdot v).$$

Because $v \cdot v > 0$, we conclude

$$\lambda = \overline{\lambda},$$

which means λ is real.

(b) If $\mu, \nu \in \mathbb{R}$ are distinct eigenvalues for $T\colon V \to V$, then $E_\mu \cap E_\nu = \{0\}$.

Solution: TRUE.
Let $v \in E_\mu \cap E_\nu$. Then

$$T(v) = \mu v \quad \text{and} \quad T(v) = \nu v.$$

Hence

$$\mu v = \nu v \quad \Rightarrow \quad (\mu - \nu)v = 0.$$

Since $\mu \neq \nu$, it follows that $v = 0$. Thus $E_\mu \cap E_\nu = \{0\}$.

(c) If $W \subset \mathbb{R}^n$ is a subspace of $\mathbb{R}^n$ and $x \in \mathbb{R}^n \setminus W$, then $x \in W^\perp$.

Solution: FALSE.
Take $W = \text{Span}((1,0)) \subset \mathbb{R}^2$. Then $W^\perp = \text{Span}((0,1))$.
Let $x = (1,1)$. Then $x \notin W$ (it is not a multiple of $(1,0)$), but

$$x \cdot (1,0) = 1 \neq 0,$$

so x is not orthogonal to W and hence $x \notin W^\perp$. This is a counterexample.

(d) Suppose $\mu, \nu \in \mathbb{R}$ are distinct eigenvalues for $T\colon V \to V$. If $v_\mu \in E_\mu$ and $v_\nu \in E_\nu$, then v_μ and v_ν are linearly independent.

Solution: TRUE.
Assume $v_\mu \neq 0$, $v_\nu \neq 0$ and

$$T(v_\mu) = \mu v_\mu, \quad T(v_\nu) = \nu v_\nu, \quad \mu \neq \nu.$$

Suppose $av_\mu + bv_\nu = 0$ for some scalars a, b. Apply T:

$$T(av_\mu + bv_\nu) = aT(v_\mu) + bT(v_\nu) = a\mu v_\mu + b\nu v_\nu = 0.$$

We now have

$$\begin{cases} av_\mu + bv_\nu = 0, \\ a\mu v_\mu + b\nu v_\nu = 0. \end{cases}$$

Multiply the first equation by ν and subtract:

$$(a\mu v_\mu + b\nu v_\nu) - (a\nu v_\mu + b\nu v_\nu) = a(\mu - \nu)v_\mu = 0.$$

Since $\mu \neq \nu$ and $v_\mu \neq 0$, we get $a = 0$. Then $bv_\nu = 0$ in the first equation gives $b = 0$. Thus the only linear combination giving 0 is the trivial one, so v_μ and v_ν are linearly independent.